International Journal of Theoretical Physics, Vol. 34, No. 12, 1995

Geometric and Analytical Aspects of Anyons

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Received February 6, 1995

We consider quantum systems of n indistinguishable spinless particles, constrained to closed compact surfaces and satisfying fractional statistics (anyons). We question the traditional choice of a configuration space, and show that a theory maintaining the diagonal is possible. Such a theory leads naturally to questions in algebraic geometry involving desingularizations of certain algebraic varieties. The desingularizations induce the possibility of an "exotic exclusion principle" for anyons, wherein multiple occupancy is not excluded in general.

1. THE STANDARD THEORY

1.1. The Classical n-Body Configuration Space

In studying the kinematics of n-body systems in classical mechanics one traditionally identifies the coordinates of the n-body system in physical space with the coordinates of a single virtual object in so-called configuration space, which is in general a subvariety of the higher dimensional space R^{3n}. The quantum theory of anyons, quasiparticles in two-dimensional space, began with an analogous step (Wilczek, 1990).

We consider such a theory of systems of $n \geq 2$ indistinguishable spinless particles constrained to a compact Riemann surface M; unless explicitly stated, M will be assumed to be without boundary. It has become standard to work with the configuration space $C_n(M)$ of such a system defined as follows (Birman, 1975). Let M^n be the n-fold Cartesian product of M with itself, and let δ_n denote the subset of M^n consisting of all points where two or more particle positions coincide (i.e., the diagonal). Then

$$C_n(M) = (M^n \backslash \delta_n)/S_n \tag{1.1}$$

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0020-7748/95/1200-2435$07.50/0

where S_n denotes the group of permutations of n symbols, the slash (/) designates the formation of a quotient modulo S_n, and the backslash (\) designates the deletion of the subset δ_n.

The space $C_n(M)$ has been studied extensively in algebraic topology in connection with the theory of braids and knots and also in connection with the mapping class groups of Riemann surfaces. The fundamental group of $C_n(M)$ is precisely the n-braid group $B_n(M)$ of M (Birman, 1975). Given a Riemannian metric on M, $C_n(M)$ inherits the corresponding product metric which lifts to a Riemannian metric on the universal covering space $\tilde{C}_n(M)$ of $C_n(M)$. The deck transformations, i.e., elements of $B_n(M)$, are a set of isometries of $\tilde{C}_n(M)$.

More recently the theory of anyons has brought about further study of $C_n(M)$, this time involving also geometric analysis, in particular a theory of twisted Laplacians on $C_n(M)$ (Baker *et al.*, 1993), which occur in the theory as follows.

Let χ_k: $B_n(M) \to \mathrm{U}(k)$ be an irreducible unitary representation, and let $E(\chi_k)$ be the canonical flat vector bundle over $C_n(M)$ associated with this representation (Milnor, 1958). It has been proposed in Imbo *et al.* (1990), for example, that natural candidates for stationary wave functions for the system of n indistinguishable quantum particles on M, satisfying fractional statistics, are smooth sections of the bundle $E(\chi_k)$ or, equivalently, smooth, χ_k-equivariant, $\mathbb{C}^k$-valued functions on $\tilde{C}_n(M)$.

The statistics or quantization is determined by the choice of representation χ_k, and in this context, the phenomenon of phase changes of the wave function on interchanging particle positions translates as the χ_k-equivariance. A $\mathbb{C}^k$-valued function $\tilde{\psi}$ on $\tilde{C}_n(M)$ is χ_k-equivariant provided

$$\tilde{\psi}(gz) = \chi_k(g)\tilde{\psi}(z) \tag{1.2}$$

$$\text{for all} \quad z \in \tilde{C}_n(M) \quad \text{and for all} \quad g \in B_n(M)$$

Henceforth, we write the equivariance condition as $\tilde{\psi} \circ g = \chi_k(g)\tilde{\psi}$. In the Schrödinger theory one requires that the wave functions satisfy a Schrödinger wave equation. For the stationary wave functions this boils down to a spectral problem

$$\tilde{\Delta}\tilde{\psi} = \lambda\tilde{\psi} \tag{1.3}$$

where $\tilde{\Delta}$ is the Laplacian on $\tilde{C}_n(M)$, the lift of the $\bar{\partial}$ Laplacian on $C_n(M)$.

Observe that from a function-theoretic viewpoint all of the above theory can, and in fact only needs to be carried out on the covering $\tilde{C}_n(M, \chi_k)$ of $C_n(M)$ corresponding to the normal subgroup $\mathrm{Ker}(\chi_k)$ of $B_n(M)$. Here the group of deck transformations is just the image of χ_k. In such a program, a system of n-indistinguishable bosons corresponds to the trivial U(1) representation and that of fermions corresponds to the U(1) representation onto the subgroup $\{-1, 1\}$.

The adoption of (1.1) as the configuration space for a quantum theory of particles obeying fractional statistics seems to have been institutionalized through a sequence of articles beginning with Laidlaw and DeWitt (1971) and Leinaas and Myrheim (1977). It can be seen throughout almost all subsequent work, even in theories which were independently developed (Wilczek, 1990). The exclusion of the diagonal δ_n in (1.1) by Leinaas and Myrheim seemed somewhat arbitrary. However, subsequently, a theoretical justification for the choice of $C_n(M)$ as configuration space has been put forward by Goldin *et al.* (1980, 1981, 1983; Goldin and Sharp, 1983, 1991).

The Schrödinger picture outlined above, culminating in (1.2), (1.3), proposes that one works with wave functions which are multivalued on the configuration space $C_n(M)$ and single valued on the cover $\tilde{C}_n(M, \chi_k)$. This is an example of a general program called "quantization on multiply connected spaces." With this as background, we raise certain questions concerning the choice of $C_n(M)$ as configuration space, plus questions surrounding the resulting Schrödinger picture, (1.2), (1.3). The questions are aimed simultaneously at the mathematical difficulties as well as the physical consistency of the resulting theory.

1.2. Questions Surrounding the Configuration Space

The ability to solve the Schrödinger wave equation gives a program for analyzing the quantum theory of anyons, advocated, for example, in Imbo *et al.* (1990). There are, however, serious mathematical and physical questions to be addressed in that program.

First, the choice of the configuration space $C_n(M)$ has the immediate consequence that the particles must satisfy an exclusion principle wherein no two particles can occupy the same position. Now, multiple occupancy is possible for bosons. Thus, since anyons are particles conjectured to have statistics which interpolate between those of bosons and fermions, it would seem reasonable to consider, at least in the case of U(1) representations, that the exclusion principle satisfied by anyons might also interpolate between that for bosons and that for fermions. From this standpoint, the exclusion of the diagonal δ_n from the configuration space is a restrictive assumption in the theory and seems to be an undesirable starting point.

Our second set of questions has also to do with the exclusion of the diagonal. The study of the wave equation for the multivalued wave functions on $C_n(M)$ leads to considering a spectral problem (1.3) for a twisted Laplacian on $C_n(M)$ (Baker *et al.*, 1993). For our present purposes, it suffices to restrict consideration to the case of cyclic U(1) representations.

Here we see that for each of the extremal cases $\theta = 0$ (bosons) and $\theta = \pi$ (fermions) the Hamiltonian possesses a pure point spectrum, due to the assumed compactness of the surface M. The complete system of eigenfunctions of the scalar Laplacian on M orthonormal in $L^2(M)$ corresponds to the set of pure energy states of the one-particle problem, i.e., $n = 1$. For a system of n bosons, the complete basis of orthonormal eigenfunctions on $L^2(M^n/S_n)$ is obtained by taking symmetrized products of n one-particle eigenstates: an elementary procedure analogous to "separation of variables." For a system of n fermions, one obtains a complete basis of antisymmetric eigenfunctions on M^n/S_n, and thus on $C_n(M)$, by taking antisymmetric products of n one-particle eigenstates. These are the only two U(1) representations for which the exact energy levels and the corresponding wave functions are known for ideal anyons (i.e., no interactions). Even for a system of as few as three ideal anyons on a compact surface it is not known, in the intermediate case $0 < \theta < \pi$, whether the Hamiltonian has a pure point spectrum.

In general, the fact that the configuration space $C_n(M)$ is noncompact and noncomplete renders questions surrounding the spectral problem quite complicated. The difficulties become apparent in the simplest case, treated in Baker *et al.* (1993); we briefly describe these results.

We choose M in this case to be a bounded domain with smooth boundary ∂M in the Euclidean plane $\mathbb{E}^2$. In the coordinate system on $\mathbb{C}^n$ we denote a generic point by $z = (z_1, \ldots, z_n)$, $z_i \in \mathbb{C}$, and $z^* = (\bar{z}_1, \ldots, \bar{z}_n)$ its conjugate.

The Euclidean Laplacian Δ acting on $C^\infty(\mathbb{C}^n)$ is given by

$$\Delta f = 4 \sum_{k=1}^{n} \partial_k \bar{\partial}_k f \tag{1.4}$$

where $\partial_k f = \partial f/\partial z_k$ and $\bar{\partial}_k f = \partial f/\partial \bar{z}_k$. The diagonal is then given by

$$\delta_n = \{z = (z_1, \ldots, z_n) \in M^n\colon z_i = z_j \text{ for some } i \neq j\}$$

If $\chi\colon \pi_1(C_n(M)) \to \mathrm{U}(1)$ is a finite, one-dimensional, irreducible representation, it corresponds to the choice of an integer $m \geq 1$, with the corresponding cyclic group $G_m = \{\exp(2\pi ik/m), k = 0, 1, \ldots, m - 1\}$ consisting of the m roots of unity.

Let $V\colon C_n(M) \to \mathbb{R}$ be a real-valued function, $V \in C^\infty(C_n(M))$, and satisfies

$$V(\sigma z, \sigma z^*) = V(z, z^*), \qquad z \in (M^n \backslash \delta_n), \qquad \text{for all} \quad \sigma \in S_n \tag{1.5}$$

V will play the role of a symmetric potential for the system of anyons. Let $\tilde{V}\colon \tilde{C}_n(M, \chi) \to \mathbb{R}$ denote the lift of V to the covering space $\tilde{C}_n(M, \chi)$.

We seek eigenvalues $\lambda \in \mathbb{C}$ and eigenfunctions $\tilde{\psi}\colon \tilde{C}_n(M, \chi) \to \mathbb{C}$ which satisfy the Schrödinger equation,

$$-\tilde{\Delta}\tilde{\psi} + \tilde{V}\tilde{\psi} = \lambda\tilde{\psi} \qquad \text{on} \quad \tilde{C}_n(M, \chi) \tag{1.6}$$

$$\tilde{\psi} = 0 \qquad \text{on} \quad \tilde{\Gamma}_n \tag{1.7}$$

where $\tilde{\Gamma}_n$ is the lift of the boundary $\Gamma_n = \partial M^n$. Equation (1.6) is merely (1.3) with a potential.

We also require that $\tilde{\psi}$ satisfy the fractional statistics corresponding to the choice of representation χ, and so (1.2) becomes

$$\tilde{\psi}(\gamma z, \gamma z^*) = \chi(\gamma)\tilde{\psi}(z, z^*) \qquad \text{for all} \quad \gamma \in B_n(M) \tag{1.8}$$

G_m-Equivariant functions on $C_n(M)$ satisfying (1.8) are all naturally constructable using the discriminant. Let φ: $\mathbb{C}^n \to \mathbb{C}$ be defined by

$$\varphi(z_1, \ldots, z_n) = \prod_{i<j} (z_i - z_j) \tag{1.9}$$

Let φ_m: $\tilde{C}_n(M, \chi) \to \mathbb{C}$ denote any one of the m roots of the discriminant $[\varphi(z)]^2$ according to $\varphi_m(z) = [\varphi(z)]^{2/m}$. Since φ^2 never vanishes on $\tilde{C}_n(M, \chi)$, given any equivariant $\tilde{\psi}$ satisfying (1.2), we may consider the map

$$\tilde{f}\colon \tilde{C}_n(M, \chi) \to \mathbb{C} \text{ given by } \tilde{f}(z, z^*) = \tilde{\psi}(z, z^*)/\varphi_m(z)$$

Clearly, $\tilde{f}$ is an invariant function, $\tilde{f}(\sigma z, \sigma z^*) = \tilde{f}(z, z^*)$ for all $\sigma \in B_n(M)$, for $z \in \tilde{C}_n(M, \chi)$, and thus corresponds to an invariant function F: $C_n(M) \to \mathbb{C}$ defined by

$$\tilde{f}(z, z^*) = (F \circ \pi)(z, z^*)$$

where π: $\tilde{C}_n(M, \chi) \to C_n(M)$ is the natural projection.

Now setting $\alpha = 2/m$, a simple computation using (1.4)–(1.9) shows that

$$\tilde{\psi} = (\varphi_m)(F \circ \pi)\colon \tilde{C}_n(M, \chi) \to \mathbb{C}$$

is an eigenfunction with eigenvalue $\lambda \in \mathbb{C}$, satisfying (1.6)–(1.8), if and only if F: $(M^n \backslash \delta_n) \to \mathbb{C}$ satisfies equivalently

$$-\Delta F - \frac{4\alpha}{\varphi} \sum_{k=1}^{n} \frac{\partial \varphi}{\partial z_k} \frac{\partial F}{\partial \bar{z}_k} + VF = \lambda F \qquad \text{on} \quad (M^n \backslash \delta_n) \tag{1.10}$$

$$F = 0 \qquad \text{on} \quad \Gamma_n \tag{1.11}$$

$$F(\sigma z, \sigma z^*) = F(z, z^*) \qquad \text{for all} \quad \sigma \in S_n \tag{1.12}$$

The peculiarities of the more transparent spectral problem (1.10)–(1.12) lie in the fact that the space $(M^n \backslash \delta_n)$ is noncompact and incomplete. In particular, the subset δ_n of real codimension 2 has been deleted from M^n. Furthermore, from (1.9), the coefficients of the elliptic operator appearing in (1.10) become singular on neighborhoods of the diagonal δ_n. These singularities are inherited from the problem on the covering space $\tilde{C}_n(M, \chi)$;

specifically, because of (1.8), the group action of $B_n(M)$ has fixed points on the diagonal δ_n. The singularities are in this sense inherent to the problem (1.3) or (1.6), by the deletion of δ_n.

In Baker *et al.* (1993) it is proved that if V is a hard-core repulsive potential of a certain type, then (1.10)–(1.12) possess a pure point spectrum of real eigenvalues and a corresponding set of equivariant eigenfunctions satisfying (1.6)–(1.8) which form an orthonormal and complete set in an appropriate Hilbert space. These eigenfunctions vanish on the lift of the diagonal $\tilde{\delta}_n$ and decay exponentially to zero on neighborhoods of $\tilde{\delta}_n$. In a certain sense the interaction via the potential V compensates for the noncompactness and incompleteness of $C_n(M)$.

Loss and Fu (1991) have proposed that without the interaction of a hard-core repulsive potential, the Schrödinger picture is not properly posed. Their conclusions are based on computations and analysis of virial coefficients for both an interacting anyon gas with hard-core potential and for a noninteracting anyon gas. The spectral problem for the case without such a potential, i.e., $V = 0$, $m \geq 3$, is completely open.

Although these questions pose a very interesting line of investigation, it is of course also natural to ponder a theory which uses a configuration space other than $C_n(M)$, for example, one where the configuration space is the entire symmetric product $S_n(M) = M^n/S_n$. This is the central issue of the present article.

2. GENERALIZED MULTIVALUED WAVE FUNCTIONS

2.1. A Generalized Configuration Space and Its Branched Cover

Throughout the rest of the paper, for concreteness, we shall consider only irreducible U(1) representations of $B_n(M)$. In such cases, since the surface M is compact, it is well known (Thouless and Wu, 1985) that if M has genus $g \geq 1$, then $B_n(M)$ admits only two representations, the trivial one corresponding to bosons and that corresponding to $\{1, -1\}$, i.e., fermions. These two cases have already been discussed in Section 1.2. Thus, without any loss, we may take M to be the 2-sphere S^2. We explore the possibilities of a theory which possesses at least the following properties:

(i) The configuration space should not dictate *a priori* that anyons satisfy the same exclusion principle as do fermions, as in the choice of $C_n(M)$. The precise exclusion principle for anyons should be deduced from the general theory as opposed to being imposed.

(ii) The Schrödinger picture should be amenable to analysis: in particular the associated spectral problem for the Hamiltonian should be tractable.

In order to achieve this, we reexamine an assumption of the standard theory. The fact that the phenomenon of fractional statistics was first represented using a multiply connected configuration space has somehow led to the idea that a multiply connected configuration space is indispensable for fractional statistics. Our point of view is more general; what is in fact indispensable in the theory is the existence of a domain for the wave function on which it is single valued together with a corresponding base or configuration space on which it will be multivalued. These we refer to the *salient features of the theory.* They are indeed consequences of the choice of $C_n(M)$ as configuration space. However, starting with these features, we wish to show that they can be realized with the choice of a more general configuration space. Thus, the relevant questions are:

(iii) Does there exist a *canonical* choice of a compact, complete, smooth domain for our wave functions, i.e., a space where the wave functions are single valued? A compact, complete space has advantages for the spectral problem. Having chosen a domain and base configuration space on which wave functions are multivalued, to what extent does this domain determine the quantum physics of anyon interaction, for example, an exclusion principle?

Our theory described below provides answers to the issues (i)–(iii) for the case $k = 1$ with $S_n(M)$ as configuration space. It points toward the possibility of what we refer to as an "exotic exclusion principle" for anyons on S^2 wherein multiple occupancy is not excluded in general. Our wave functions are constructed from solutions of Schrödinger equations on compact smooth projective varieties, as opposed to the inherently singular elliptic equations on noncompact incomplete manifolds encountered in the standard theory. They are referred to as "generalized multivalued wavefunctions."

We thus begin with the choice of the symmetric product $S_n(M)$ as configuration space. When $M = S^2$ is thought of as the complex projective line, the space $S_n(M)$ is naturally identified with the n-dimensional complex projective space $\mathbf{CP}^n$ (Griffiths and Harris, 1978). Under this identification the diagonal δ_n gets identified with the "discriminant hypersurface" $\Delta_n = 0$. The next section is devoted to such details.

It is well known that the U(1) representations of $B_n(M)$ are the homomorphisms onto the cyclic groups G_m of the mth roots of unity, where m is a positive integer dividing $2(n - 1)$ (Thouless and Wu, 1985). From the function-theoretic perspective outlined above, the physics of multivalued wave functions is captured by the geometry of the cyclic coverings of the configuration space. With $C_n(M)$ as the configuration space, such cyclic coverings exist and are n-dimensional complex manifolds. These are open subsets of branched coverings $X_n(M, m)$ of $\mathbf{CP}^n$, branched along the discriminant hypersurface, with cyclic Galois groups G_m. Note that unlike $C_n(M)$, our space $S_n(M)$ is simply connected. In addition, by Picard's theorem (Abhyankar, 1959), the spaces $X_n(M, m)$ are also simply connected. However, in general, $X_n(M, m)$ is not a manifold; it has a set of singularities of complex codimension 2 consisting of the points lying above the singular points of the discriminant hypersurface.

In this setting a system of n indistinguishable bosons on S^2 corresponds to single-valued wave functions on $\mathbf{CP}^n$. Systems of n indistinguishable fermions correspond to double-valued wave functions on $\mathbf{CP}^n$, i.e., single-valued functions on $X_n(M, 2)$.

Since $X_n(M, m)$ have singularities, at least for $n \geq 3$, there are formidable obstructions to formulating analytical problems on $X_n(M, m)$, such as the sought-after Schrödinger theory with G_m-equivariant functions on $X_n(M, m)$. We propose instead what would seem from a mathematical viewpoint to be the next best thing, to choose as a domain of our wave functions a "canonical" desingularization of $X_n(M, m)$ embedded in complex projective space. These desingularizations of $X_n(M, m)$ naturally carry an induced G_m-action. There is by now a well-established "theme" in the study of singular varieties, initiated by MacPherson (1984), which attempts to obtain information on a singular variety by studying analytical and topological data of desingularizations of the variety (Fox, Haskell, and Pardon, 1988).

Without any loss, we can focus attention on the case $m = 2(n - 1)$. Henceforth, for simplicity, we denote the space $X_n(M, 2(n - 1))$ by $\mathbf{X}^{(n)}$ and $G_{2(n-1)}$ by $\mathbf{G}$. In the next section we construct explicitly canonical desingularizations of spaces $\mathbf{X}^{(n)}$ for small values of n. Although by the grand desingularization theorem of Hironaka (1964) the existence of a desingularization of $\mathbf{X}^{(n)}$ is guaranteed, the problem of constructing a canonical desingularization, for general n, seems to be unresolved. In Section 3 we present the computations for canonical desingularizations for the cases $n \leq 4$. Our computations show that canonical desingularizations are computable for small n, but a method which would produce a canonical desingularization for any n is yet to be developed. Nonetheless, our computations for $n \leq 4$ leave us optimistic; we leave the general question for the future.

Recently, the problem of *explicitly* resolving the singularities of symmetric products has been raised in a different context by Fulton and McPherson (1994). In general that problem also remains open.

2.2. An Exotic Exclusion Principle

We proceed to discuss the consequences of our choice of $S_n(M)$ as the configuration space, in particular, some essentials of a Schrödinger theory.

Let π denote the projection of $\mathbf{X}^{(n)}$ onto $\mathbf{CP}^n$. We let $D_n \subset \mathbf{CP}^n$ denote the discriminant hypersurface, given by the equation $\Delta_n = 0$, and we set

$$R_n = \pi^{-1}(D_n)$$

Also, let σ_n denote the singular locus of D_n and $\Sigma_n \subset \pi^{-1}(\sigma_n)$ be the singular locus of $\mathbf{X}^{(n)}$.

Let $\mathbf{X}_*^{(n)}$ be a desingularization of $\mathbf{X}^{(n)}$ (not necessarily a canonical one) with the associated birational map θ from $\mathbf{X}_*^{(n)}$ to $\mathbf{X}^{(n)}$. Since for $n = 2$, $\mathbf{X}^{(2)}$ is itself nonsingular, we shall tacitly assume $\mathbf{X}_*^{(2)} = \mathbf{X}^{(2)}$. Henceforth we focus on $n \geq 3$. The open set $\mathbf{X}_*^{(n)} - \theta^{-1}(\Sigma_n)$ is isomorphic via θ to the set $\mathbf{X}^{(n)} - \Sigma_n$. The closure of $R_n - \Sigma_n$ in $\mathbf{X}_*^{(n)}$ is called the proper transform of R_n in $\mathbf{X}_*^{(n)}$ and we denote it by $\mathbf{T}^{(n)}$.

The Galois group $\mathbf{G}$ acts naturally on $\mathbf{X}_*^{(n)}$. We shall use explicitly the fact that for all $y \in \mathbf{T}^{(n)}$, the orbit of y under the action of $\mathbf{G}$, i.e., the set $\{gy \mid g \in \mathbf{G}\}$, has cardinality strictly less than that of $\mathbf{G}$. In other words, given $y \in \mathbf{T}^{(n)}$, there exists $g \in \mathbf{G}$, depending on y, such that g is not the identity and $gy = y$.

$\mathbf{X}_*^{(n)}$ will be a smooth (closed) projective variety and hence can be viewed as a complex n-dimensional Kähler manifold. Since $\mathbf{G}$ is a finite group, every choice of metric on $\mathbf{X}_*^{(n)}$ yields a corresponding $\mathbf{G}$-invariant metric on $\mathbf{X}_*^{(n)}$. When $\mathbf{X}_*^{(n)}$ is a canonical embedded desingularization, it is accompanied by a natural choice of a Kähler metric, namely the one induced by the embedding of $\mathbf{X}_*^{(n)}$ into $\mathbf{CP}^{n+1+k}$ which is itself equipped with the Fubini–Study metric (Griffiths and Harris, 1978). There is of course no unique embedding of $\mathbf{X}_*^{(n)}$ into higher dimensional projective spaces; however, it is well known that the metrics induced from different embeddings are quasi-isometric (Fox, Haskell, and Pardon, 1988).

With this setting it is natural to study the singular variety $\mathbf{X}^{(n)}$ through a desingularization $\mathbf{X}_*^{(n)} \to \mathbf{X}^{(n)}$, as they are intricately related. A flavor of this is given by the following. It follows from a theorem of Pardon and Stern (1991) that the $L^2 - \bar{\partial}$ cohomology groups of $\mathbf{X}^{(n)} - \Sigma_n$, with Dirichlet boundary conditions, $H_{\mathrm{D}}^{0,q}(\mathbf{X}^{(n)} - \Sigma_n)$, are birational invariants of $\mathbf{X}^{(n)}$. In fact we have

$$H_{\mathrm{D}}^{0,q}(\mathbf{X}^{(n)} - \Sigma_n) \cong H^{0,q}(\mathbf{X}_*^{(n)}), \qquad 0 \leq q \leq n \tag{2.1}$$

where $\mathbf{X}^{(n)} \to \mathbf{X}_*^{(n)}$ is any resolution of singularities of $\mathbf{X}^{(n)}$ and $H^{0,q}(\mathbf{X}_*^{(n)})$ is the usual Dolbeault cohomology group of the smooth variety $\mathbf{X}_*^{(n)}$. As a consequence of (2.1), the arithmetic genus extends as a birational invariant,

$$\mathcal{X}_{\mathrm{D}}(\mathbf{X}^{(n)} - \Sigma_n) = \sum_{q=0}^{n} (-1)^q H_{\mathrm{D}}^{0,q}(\mathbf{X}^{(n)} - \Sigma_n) = \mathcal{X}(\mathbf{X}_*^{(n)}) \tag{2.2}$$

In the most general case we may regard $\mathbf{X}_*^{(n)}$ as being equipped with a $\mathbf{G}$-invariant Riemannian metric. Let Δ denote the corresponding Laplacian acting on the linear space C^∞_* of smooth **G**-equivariant complex-valued functions on $\mathbf{X}^{(n)}_*$.

In this setting it is automatic (Griffiths and Harris, 1978) that Δ has a unique extension to $L^2(C^\infty_*)$ with a pure point spectrum of nonnegative eigenvalues, listed according to multiplicities

$$0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

The corresponding set of $L^2(C^\infty_*)$ orthonormalized eigenfunctions $\{\Psi_j\}$ satisfy

$$\Delta\Psi_j = \lambda_j\Psi_j, \qquad j = 1, 2, \ldots$$

where $j = 1, 2, \ldots$, and form a complete orthonormal system in $L^2(C^\infty_*)$. It is clear that the **G**-equivariance of the eigenfunctions forces Ψ_j to vanish on $\mathbf{T}^{(n)}$.

We let ψ_j denote the restriction of Ψ_j to the open set $\mathbf{X}^{(n)}_* - \theta^{-1}(\Sigma_n)$. By composition,

$$\phi_j = \psi_j \circ \theta^{-1}: \qquad \mathbf{X}^{(n)} - \Sigma_n \to \mathbf{C}$$

is a smooth **G**-equivariant complex-valued function. It is clear that ϕ_j vanishes on $R_n - \Sigma_n$. The system $\Phi = \{\phi_j\}$ is complete and orthonormal in $L^2(\mathbf{X}^{(n)})$ with respect to the measure induced on $\mathbf{X}^{(n)}$ via θ.

Now since $\Phi \subset C^\infty(\mathbf{X}^{(n)} - \Sigma_n)$, the eigenfunctions are extendible to functions on $\mathbf{X}^{(n)}$, but they will not in general be continuous on $\mathbf{X}^{(n)}$. Function values at points in δ_n will depend on the stratum in $\tilde{\delta}_n$ within which the point is approached. We interpret the family Φ to be the Schrödinger system of generalized multivalued wave functions on $S_n(M)$.

The antisymmetric wave functions for fermions discussed in Section 2.1 vanish on δ_n. This fact is traditionally interpreted as the exclusion principle which prevents any two fermions from occupying the same spatial position (Canright and Johnson, 1994). We shall follow this tradition in observing the properties of our generalized wave functions.

If such wave functions are used for a quantum theory, then as a consequence the fact that the members of Φ vanish on $R_n - \Sigma_n$ can be interpreted as the preclusion of certain possible anyon configurations. Specifically, the possibility of *exactly two* anyons occupying the same position is excluded, but no other configuration is *a priori* excluded. Said in another way, anyons are permitted to coalesce in either two or more pairs at a time or in groups of three or more at a time. This is what we have already referred to as an "exotic exclusion principle" for anyons. It is our belief that the rules for occupancy or the exclusion principle for anyons are related to the nature of the singularities of the cover $\mathbf{X}^{(n)}$.

We note that, strictly speaking, the Pauli exclusion principle states that two particles cannot possess the same set of quantum numbers. What we refer to in this paper as an "exotic exclusion principle" refers merely to the question of occupancy of spatial positions. For example, two bosons can occupy the same spatial position, but fermions may not: the question as to the precise rules for anyons remains open, but in the view of some physicists, it is appropriate to refer to the rules for occupancy of spatial positions as a type of "exclusion principle" (Haldane, 1991; Canright and Johnson, 1994). However, rules for occupancy of spatial positions are not to be assumed equivalent to the more encompassing Pauli exclusion principle.

3. CANONICAL DESINGULARIZATIONS

In this section we present the computations which produce canonical desingularizations of the singular cover $\mathbf{X}^{(n)}$ for $n \leq 4$. These examples are instructive and demonstrate that canonical desingularizations are computable for small n. A general procedure for computing canonical desingularizations for general n is yet to be worked out.

By $\mathbf{CP}^n$ we denote the n-dimensional complex projective space. Let $\mathbf{V}$ be the $\mathbf{C}$-vector space of binary forms of degree n with coefficients in $\mathbf{C}$ (the field of complex numbers). $\mathbf{CP}^n$ can be identified with $\mathbf{P(V)}$ via the correspondence which associates to a point with homogeneous coordinates $(a_0, \ldots, a_n)$ the binary form

$$F = a_0 X^n + \cdots + \binom{n}{r} a_r X^{n-r} Y^r + \cdots + a_n Y^n$$

For positive integers d, e with $de \leq n$, let $W(n; d, e)$ be the subset of $\mathbf{V}$ consisting of those forms which are divisible by the eth power of a binary form of degree d. It follows that $W(n; p, q) \subseteq W(n; r, s)$ whenever $rs \leq pq$ and $s \leq q$. Corresponding to $W(n; d, e)$ there is an irreducible rational subvariety of $\mathbf{CP}^n$; we denote this subvariety also by $W(n; d, e)$, since no confusion seems likely. We shall tacitly assume $n \geq 2$.

Observe that $W(n; 1, 2)$ is the discriminant hypersurface and $W(n; 1, n)$ is the rational normal curve of degree n in $\mathbf{CP}^n$. For $n \geq 3$, the discriminant hypersurface has singularities; its singular locus is the union of all the subvarieties $W(n; d, e)$ with $e \geq 2$ and $d + e \geq 4$.

Let Δ_n denote the discriminant of F. Note that Δ_n is unique up to a scalar factor, and it is an irreducible homogeneous polynomial in $a_0, \ldots, a_n$ of degree $2(n - 1)$. The $2(n - 1)$-cyclic cover of $\mathbf{CP}^n$ ramified along the discriminant hypersurface is the hypersurface $\mathbf{X}^{(n)}$ in $\mathbf{CP}^{n+1}$ defined by

$$Z^{2(n-1)} - \Delta_n = 0$$

Here $\mathbf{X}^{(n)}$ is a normal hypersurface and its singularities lie exactly above the singularities of $\Delta_n = 0$. In the following we shall describe a canonical embedded desingularization of $\mathbf{X}^{(n)}$ when n is either 3 or 4.

It is necessary to set up some notation. For an irreducible subvariety N of a variety M let $\mu(M, N)$ denote the multiplicity of M at the generic point of N. Let $m(M)$ be the maximum of $\mu(M, N)$ where N ranges over all the irreducible subvarieties. By the maximum-multiplicity locus (henceforth abbreviated as *mm*-locus) of M we mean the union of all irreducible subvarieties N of M such that $\mu(M, N) = m(M)$. A pair (L^*, M^*) is said to be the canonical transform of a pair (L, M) of varieties if M is a (closed) subvariety of L, L^* is the blowup of L along the *mm*-locus of M, and M^* is the proper transform of M in L^*. We make the convention that $L^* = L$ and $M^* = M$ in case M is nonsingular.

Consider the sequence $(\mathbf{Y}_i, \mathbf{X}_i)$ where $0 \leq i$, $\mathbf{Y}_0 = \mathbf{CP}^{n+1}$, $\mathbf{X}_0 = \mathbf{X}^{(n)}$, and $(\mathbf{Y}_{i+1}, \mathbf{X}_{i+1})$ is the canonical transform of $(\mathbf{Y}_i, \mathbf{X}_i)$. Now we may ask: Does there exist an integer r such that (1) $\mathbf{X}_r$ is nonsingular? (2) $\mathbf{X}_r$ and $\mathbf{Y}_r$ both are nonsingular? The answers to these questions are not known in general; but when $n = 2$, 3, or 4 we can easily verify that they are in the affirmative. When $n = 2$ our hypersurface $\mathbf{X}^{(n)}$ is nonsingular and hence we may take $r = 0$. Below we shall discuss the cases $n = 3$, 4 in a little more detail. For the basic definitions and facts concerning the operation of 'blowing-up' the reader may refer to Abhyankar (1966) or Hartshorne (1977).

3.1. The Case of $n = 3$

We rename the homogeneous coordinates (a_0, a_1, a_2, a_3) as (a, b, c, d). Then, up to a numerical factor,

$$\Delta_3 = (ad - bc)^2 - 4(ac - b^2)(bd - c^2)$$

is the discriminant. The singular locus of the discriminant surface $\Delta_3 = 0$ is the 'twisted cubic curve' R: $(ad - bc)^2 = (ac - b^2) = (bd - c^2) = 0$. It is easy to verify that every point of R is a double-point of the discriminant surface and R is the *mm*-locus. The singular locus of $\mathbf{X}_0$: $Z^4 - \Delta_3 = 0$ is the curve C: $Z = (ad - bc)^2 = (ac - b^2) = (bd - c^2) = 0$. It is also the *mm*-locus. Since C is nonsingular, the blowup $\mathbf{Y}_1$ of $\mathbf{Y}_0 = \mathbf{CP}^4$ along C is also nonsingular. Let E denote the exceptional locus of this transformation. Let S_1 be the proper transform of the surface S in $\mathbf{Y}_0$ defined by

$$(ad - bc)^2 = (ac - b^2) = (bd - c^2) = 0$$

(the cylinder over R). Then, $m(\mathbf{X}_1) = 2$ and the singular locus of $\mathbf{X}_1$ (= the *mm*-locus of $\mathbf{X}_1$) is the curve $\mathbf{X}_1 \cap E \cap S_1$. Again, $\mathbf{X}_1 \cap E \cap S_1$ is nonsingular and hence so is $\mathbf{Y}_2$. Furthermore, $\mathbf{X}_2$ is nonsingular.

3.2. The Case of $n = 4$

This case seems to have all the essential features of the general problem. Therefore in comparison to the cubic case it is somewhat more complicated.

Rename the homogeneous coordinates $(a_0, \ldots, a_4)$ as (a, b, c, d, e). Then, up to a numerical factor the discriminant Δ_4 is $I^3 - 27J^2$, where

$$I = ae - 4bd + 3c^2 \qquad \text{and} \qquad J = ace + 2bcd - c^3 - ad^2 - b^2e$$

are the two well-known invariants of the quartic. The singular locus of the discriminant-threefold $\Delta_4 = 0$ is the union of the two irreducible surfaces $W(4; 1, 3)$ and $W(4; 2; 2)$. The surface $W(4; 2; 2)$ is nonsingular and intersects $W(4; 1, 3)$ along the later's singular locus, namely the nonsingular (rational normal) quartic curve $W(4; 1, 4)$. We define $W(4; 1, 3)$ (set theoretically) by $I = J = 0$. The hypersurface $I = 0$ is nonsingular. A nonsingular point of $W(4; 1, 3)$ is also a nonsingular point of $J = 0$; moreover, at such a point, hypersurfaces $I = 0$ and $J = 0$ meet nontangentially. $W(4; 1, 4)$ is the *mm*-locus of the discriminant hypersurface, consisting of its triple points.

Let W_1, W_2, and C denote the subvarieties of $\mathbf{X}_0$: $Z^6 - \Delta_n = 0$ lying above $W(4; 1, 3)$, $W(4; 2; 2)$, and $W(4; 1, 4)$, respectively. Then, the singular locus of $\mathbf{X}_0$ is $W_1 \cup W_2$, $m(\mathbf{X}_0) = 3$, and the *mm*-locus of $\mathbf{X}_0$ is C. Let T_1, T_2, and S denote the cylinders over $W(4; 1, 3)$, $W(4; 2; 2)$, and $W(4; 1, 4)$, respectively. Observe that $W_j = T_j \cap \mathbf{X}_0$ for $1 \leq j \leq 2$ and $C = S \cap \mathbf{X}_0$.

To study the desingularization process we may restrict our attention to the affine subspace $\mathbf{A}$: $a \neq 0$ of $\mathbf{CP}^5$ without any loss. Here, we are taking advantage of the fact that Δ_4 is an invariant of the quartic. The affine coordinates on $\mathbf{A}$ are the ratios $(b/a, c/a, d/a, e/a, Z/a)$. Let

$$f(X) = X^4 + 4(b/a)X^3 + 6(c/a)X^2 + 4(d/a)X + (e/a)$$

and let u, v, w be determined by the equation

$$f(X - (b/a)) = X^4 + 6uX^2 + 4vX + w$$

Note that the affine coordinate ring of $\mathbf{A}$ is the polynomial ring

$$A = \mathbf{C}[b/a, c/a, d/a, e/a, Z/a] = \mathbf{C}[b^*, u, v, w, z]$$

where $b^* = b/a$ and $z = Z/a$. Polynomials

$$i = w + 3u^2 \text{ and } j = uw - u^3 - v^2$$

are the dehomogenizations of I, J. Our hypersurface $\mathbf{X}_0$ corresponds to the ideal $(z^6 - i^3 + 27j^2)A$. The surfaces $W(4; 1, 3)$, $W(4; 2, 2)$ are defined by the ideals $(i, j)A$ and $(v, w - 9u^2)A$, respectively, whereas the curve C is given by the ideal $(u, v, w)A$.

Using the above ring-theoretic description, it is straightforward to verify the following five claims.

Let $E_p \subset \mathbf{Y}_P$ denote the exceptional locus for the pth canonical transform. Also, let T_{qp}, S_p, and W_{qp} denote the proper transforms of T_q, S, and W_q in $\mathbf{Y}_P$, respectively.

(1) For the first canonical transform, we have $m(\mathbf{X}_1) = 3$ and here the mm-locus of $\mathbf{X}_1$ is the nonsingular surface $\mathbf{X}_1 \cap E_1 \cap T_{11}$.
(2) For the second canonical transform, we still have $m(\mathbf{X}_2) = 3$ and the mm-locus of $\mathbf{X}_2$ is the nonsingular surface $\mathbf{X}_2 \cap E_2 \cap T_{12}$.
(3) After the third canonical transform there is a drop in the maximum multiplicity. We have $m(\mathbf{X}_3) = 2$ and the mm-locus of $\mathbf{X}_3$ is seen to be $W_{13} \cup W_{23}$. Surfaces W_{13}, W_{12} are nonsingular, disjoint subvarieties of $\mathbf{X}_3$.
(4) Next, $m(\mathbf{X}_4) = 2$. The mm-locus of $\mathbf{X}_4$ is the union of two disjoint nonsingular varieties $\mathbf{X}_4 \cap E_4 \cap T_{14}$ and $\mathbf{X}_4 \cap E_4 \cap T_{24}$.
(5) Finally, $\mathbf{X}_5$ is nonsingular. Since the center of each of the above blowing-ups is nonsingular, the variety $\mathbf{Y}_5$ is also nonsingular.

4. REMARKS

4.1.

Goldin *et al.* (1980, 1981, 1983; Goldin and Sharp, 1983, 1991) have proposed a theoretical justification for the exclusion of the diagonal δ_n; this agrees with previous conclusions of Leinaas and Myrheim (1977) and Laidlaw and DeWitt (1971), who also argue for the exclusion of δ_n, but from different mathematical perspectives.

The solution of (1.3) for n bosons by symmetrization is a solution on M^n/S^n. For $\epsilon > 0$, let $N_\epsilon(\delta_n) = \{y \in M^n\colon d(y, \delta_n) < \epsilon\}$ be a tubular neighborhood of the diagonal, where $d(x, y)$ denotes Riemannian distance. The linear space of smooth functions

$$C_\epsilon^\infty(M^n) = \{\psi \in C^\infty(M^n)\colon \psi \,|\, N_\epsilon(\delta_n) = 0\}$$

satisfying Dirichlet boundary conditions on $\overline{N_\epsilon(\delta_n)}$, may be completed in an appropriate Sobolev space producing a Hilbert space of functions H_ϵ satisfying the above boundary conditions. The Laplacian acting on $C_\epsilon^\infty(M^n)$ or H_ϵ has a pure point spectrum of eigenvalues $\sigma_\epsilon(\Delta)$ with corresponding eigenfunctions in $C_\epsilon^\infty(M^n)$.

Using a theorem of Chavel and Feldman (1978) in weak formulation, we can show that $\sigma_\epsilon(\Delta)$ converges, as $\epsilon \to 0$, to the spectrum of the Laplacian on M^n, with matching multiplicities. Thus in this asymptotic, weak sense, we may obtain the spectrum or energy levels for bosons on M^n/S_n starting from the configuration space $C_n(M)$ without the diagonal. For nontrivial equivariance, there is no analogous result. In this sense, bosons naturally posed on M^n/S_n may be asymptotically included in the standard theory using $C_n(M)$. However, what we advocate is a theory which starts off with M^n/S_n as the configuration space.

4.2.

The idea of quasiparticles satisfying fractional statistics has been successfully used in explaining the fractional quantum Hall effect (Laughlin, 1990). There have also been attempts to explain the phenomenon of high-temperature superconductivity using anyons (Chen *et al.*, 1989; Chern *et al.*, 1991; Wilczek, 1990). This program has met with much less success. Nonetheless the theory of anyons as it unfolds presents us with interesting mathematical questions. One such is as follows.

It would be interesting to find, explicitly, nontrivial higher dimensional $\mathbb{U}(k)$, $k > 1$, unitary representations of the braid group $B_n(M)$ where M is a closed Riemann surface. Burau and Gassner representations are known for $\mathbb{R}^2$ (Jones, 1987, 1991), but even for S^2 these are not known. With such representations, it would in turn be interesting to see how much information on $B_n(M)$ is captured in spectral invariants for the configuration space using some quantization procedure for anyons, i.e., "can one hear the braid group?" This question is hinted at in Jones (1991).

4.3.

Analytical structures on the desingularization $\mathbf{X}_*^{(n)}$ were defined through a Kähler metric, ω, induced on $\mathbf{X}_*^{(n)}$ by the Fubini–Study metric on $\mathbf{CP}^{n+1+k}$ via its embedding. This is obviously convenient and natural because of the canonical nature of the Fubini–Study metric. However, strictly speaking, the physical problem would itself provide a metric on M, which yields a metric on the dense open subset $\mathbf{X}_*^{(n)} - \theta^{-1}(\Sigma_n)$ of $\mathbf{X}_*^{(n)}$; the relationship between this metric and the Kähler metric ω is another interesting issue warranting investigation.

4.4.

In concluding, we would like to emphasize that we have shown that breaking with tradition by using $S_n(M)$ as configuration space leads to a rich mathematical framework interfacing with algebraic geometry. Our branched coverings are not manifolds, since they have singularities contained in *precisely the lift of the diagonal.* It is our perspective that intricacies of the interaction physics of anyons seem related to the geometry of these singularities. Our computations show that the presence of these singularities reveals the possibility of rules for occupancy of spacial positions, which we have referred to as an "exotic exclusion principle," for anyons on S^2. Haldane (1991) has also proposed a "generalization of the Pauli principle" and Canright and Johnson (1994) have found applications of his "fractional exclusion principle." His structures and techniques are quite different from ours; in contrast, our exclusion principle has been derived from purely geometric structures.

4.5.

It is considered important in the theory of Laidlaw and DeWitt (1971) that if the space M on which the particles exist has dimension three or more, then the only possible statistics from irreducible U(1) representations are those of bosons and fermions, as a simple consequence of the topology of $C_n(M)$. It is easily checked that in our setting, with $S_n(M)$ as configuration space, the same conclusion holds.

ACKNOWLEDGMENT

The authors would like to thank professor Gerald A. Goldin for some valuable criticisms of a preliminary version of this paper.

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